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A NEW COMPUTATIONAL APPROACH FOR INVERSION OF VERY LARGE MATRICES

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Abstract—The decomposition method (Adomian[1–2]) is extended to inversion of very large matrices. Numerical examples demonstrate rapidly converging approximants in a form convenient for computer computation. Analytic representations are derived for the deterministic case. Decomposition is shown to be a new method for inversion of matrices and outside the classes of direct methods and relaxation methods.

INTRODUCTION

The decomposition method[1–3] has been applied to a wide class of linear and nonlinear, deterministic and stochastic equations as well as to matrix inversion[2, 4]. This paper considers extension to inversion of very large matrices by a combination of the decomposition method with partitioning techniques.

PART I

Consider first an elementary example

$$\Lambda = \left[\begin{array}{cc|cc} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \hline \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

partitioned into 2×2 matrices as shown. Using the procedure of references [2, 4] write $\Lambda = L + R$ where

$$L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \qquad R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \qquad L^{-1}R = \begin{bmatrix} 0 & A^{-1}B \\ D^{-1}C & 0 \end{bmatrix}$$

Now Λ^{-1} can be given in the decomposition series

$$\begin{aligned} \Lambda^{-1} &= \sum_{n=0}^{\infty} (-1)^n [L^{-1}R]^n L^{-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A^{-1}B \\ D^{-1}C & 0 \end{bmatrix}^n \cdot \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} - \begin{bmatrix} 0 & A^{-1}BD^{-1} \\ D^{-1}CA^{-1} & 0 \end{bmatrix} + \cdots \end{aligned}$$

which can be approximated as far as necessary. A two-term approximation is given by

$$\Lambda^{-1} \simeq \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$$

For larger matrices, we realize this process can be continued not only when A, B, C, D are 2×2 matrices as shown, but when the original elements λ_{ij} are matrices as well. To proceed, we consider a $2n \times 2n$ matrix for even n. Partition into

$$\Lambda = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where each element A_1, A_2, A_3, A_4 is $n \times n$. We will use the following notation. The first subscript identifies the submatrix, e.g. A_1, A_2, A_3, A_4 as above. A second subscript will identify an element of that submatrix. Thus

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} \qquad A_3 = \begin{bmatrix} A_{31} & A_{32} \\ A_{33} & A_{34} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} \qquad A_4 = \begin{bmatrix} A_{41} & A_{42} \\ A_{43} & A_{44} \end{bmatrix}$$

where now $A_{11}, A_{12}, \ldots$ are $(n/2 \times n/2)$ matrices.

The elements $A_{11}, A_{12}, \ldots$ are similarly partitioned. Thus

$$A_{11} = \begin{bmatrix} A_{111} & A_{112} \\ A_{113} & A_{114} \end{bmatrix} \qquad A_{12} = \begin{bmatrix} A_{121} & A_{122} \\ A_{123} & A_{124} \end{bmatrix}$$

etc. The process can be continued until we get scalars or possibly 2×2 matrices. We have now

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A_1^{-1}A_2 \\ A_4^{-1}A_3 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix}$$

where A_1^{-1}, A_4^{-1} can be obtained by writing

$$A_1^{-1} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{14}^{-1}A_{13} & 0 \end{bmatrix}^n \cdot \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{14}^{-1} \end{bmatrix}$$

Again, to get A_{11}^{-1}, A_{14}^{-1} we have

$$A_{11}^{-1} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A_{111}^{-1}A_{112} \\ A_{114}^{-1}A_{113} & 0 \end{bmatrix}^n \cdot \begin{bmatrix} A_{111}^{-1} & 0 \\ 0 & A_{114}^{-1} \end{bmatrix}$$

etc. proceeding with e.g. $A_{111}^{-1} \ldots 1$ is a scalar.

Suppose now we begin with $n \times n$ with odd n or with $2n \times 2n$ for odd n. The partitioning then is not symmetrical. As an example consider a 3×3 and partition as shown.

$$\Lambda = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A_1^{-1}A_2 \\ A_4^{-1}A_3 & 0 \end{bmatrix}^n \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix}$$

$$A_1^{-1} = a_{11}^{-1}$$

$$A_4^{-1} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & a_{22}^{-1}a_{23} \\ a_{33}^{-1}a_{32} & 0 \end{bmatrix}^n \cdot \begin{bmatrix} a_{22}^{-1} & 0 \\ a & a_{33}^{-1} \end{bmatrix}$$

which must converge from [2, 4] to

$$A_4^{-1} = \begin{bmatrix} \zeta/a_{33} & -a_{23}\zeta/a_{22}a_{33} \\ -a_{32}\zeta/a_{22}a_{33} & \zeta/a_{33} \end{bmatrix}$$

where $\zeta = a_{22}a_{33}/(a_{22}a_{33} - a_{23}a_{32})$.

We will assume Case I of Refs. [2, 4]* (the modification for Case II is discussed in [2, 4]). Thus we can either calculate the series for A_4^{-1} until it converges, or use the expression for the exact expression as a verification procedure (which can also be done by computing $\phi_n\Lambda$, where $\phi_n = \sum_{i=0}^{n} \Lambda_i^{-1}$ at each n, to see how closely it approaches the identity matrix. The decomposition series has the advantage that the same simple procedure is used repeatedly and that it can be extended to stochastic matrices as well as deterministic matrices. Let us consider a numerical example of a 3×3 matrice. Consider

$$\Lambda = \left[\begin{array}{c|cc} 5 & 0 & 1 \\ \hline 0 & 2 & 1 \\ 1 & 1 & 3 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n \left[\begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}\right]^n \cdot \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} 0 & A_1^{-1}A_2 \\ A_4^{-1}A_3 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix}$$

$$A_1^{-1} = 1/5$$

$$A_4^{-1} = \sum_{n=0}^{\infty} (-1)^n \left[\begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^n \cdot \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} 0 & 1/2 \\ 1/3 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$

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* $|L| > |R|$. Consider for example the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

which does not have an inverse; its determinant is 1. Now for any partitioning. $|L| = |R|$ and $|L^{-1}R| = 1$ and the technique diverges. Let A_{11} be $p \times p$, A_{12} be $p \times q$, A_{21} be $q \times p$, A_{22} be $q \times q$. Then we must verify $|L| = |A_{11}| \cdot |A_{22}| > |R| = |A_{21} \cdot A_{12}|$.

which converges to

$$A_4^{-1} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$$

(See [2, 4].) Again we can compute the series as far as necessary or write the inverse as in [2, 4]. Now $\Lambda^{-1} = \sum_{n=0}^{\infty} \Lambda_n^{-1}$ with

$$\Lambda_0^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix} = \left[\begin{array}{c|cc} 1/5 & 0 & 0 \\ \hline 0 & 3/5 & -1/5 \\ 0 & -1/5 & 2/5 \end{array}\right]$$

$$\Lambda^{-1} = (-1)\left[\begin{array}{c|cc} 1/5 & 0 & 0 \\ \hline 0 & 3/5 & -1/5 \\ 0 & -1/5 & 2/5 \end{array}\right] \cdot \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \cdot A_0^{-1}$$

$$= \begin{bmatrix} 0 & 0 & -1/5 \\ 1/5 & 0 & 0 \\ -2/5 & 0 & 0 \end{bmatrix} \cdot A_0^{-1}$$

$$= \begin{bmatrix} 0 & 1/25 & -2/25 \\ 1/25 & 0 & 0 \\ -2/25 & 0 & 0 \end{bmatrix}$$

Similarly A_2^{-1} is found by multiplying $-L^{-1}R$ in the notation of [2, 4] times A_1^{-1} thus

$$\Lambda_2^{-1} = \begin{bmatrix} 2/125 & 0 & 0 \\ 0 & 1/125 & -2/125 \\ 0 & -2/125 & 4/125 \end{bmatrix}$$

$$\vdots$$

The n-term approximation $\phi_n = \Lambda_0^{-1} + \Lambda_1^{-1} + \cdots + \Lambda_{n-1}^{-1}$ is given by

$$\phi_1 = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 3/5 & -1/5 \\ 0 & -1/5 & 2/5 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 1/5 & 1/25 & -2/25 \\ 1/25 & 3/5 & -1/5 \\ -2/25 & -1/5 & 2/5 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 27/125 & 1/25 & -2/25 \\ 1/25 & 76/125 & -27/125 \\ -2/25 & -27/125 & 54/125 \end{bmatrix}$$

$$= \begin{bmatrix} .216 & .04 & -.08 \\ .04 & .608 & -.216 \\ -.08 & -.216 & .432 \end{bmatrix}$$

Although this is only a three-term approximation easily carried much further, we note that

$$\phi_3\Lambda = \begin{bmatrix} .216 & .04 & -.08 \\ .04 & .608 & -.216 \\ -.08 & -.216 & .432 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & .01 \\ -.01 & 1 & 0 \\ .03 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. ϕ_3 is very close to the desired inverse. More accuracy is obtained by going further with ϕ_n, but this again demonstrates the rather remarkable convergence of the decomposition method as seen with a wide range of linear and nonlinear problems in [1–3] and especially [2].

PART II

Consider the matrix Λ given as

$$\Lambda = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11}, A_{12}, A_{21}, A_{22} are matrices. We write the decomposition series

$$\Lambda^{-1} = \sum_{n=0}^{\infty} \Lambda_n^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

where for Case I for example we choose

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

Then

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \qquad L^{-1}R = \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

We obtain

$$\begin{aligned} \Lambda_0^{-1} &= L^{-1} \\ \Lambda_1^{-1} &= -(L^{-1}R)\Lambda_0^{-1} = (L^{-1}R)L^{-1} \\ \Lambda_2^{-1} &= -(L^{-1}R)\Lambda_1^{-1} = (L^{-1}R)^2L^{-1} \\ \Lambda_3^{-1} &= -(L^{-1}R)\Lambda_2^{-1} = (L^{-1}R)^3L^{-1} \\ &\vdots \end{aligned}$$

$$(L^{-1}R) = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$(L^{-1}R)^2 = \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^{-1}A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

Continuing in this manner we get

$$(L^{-1}R)^3 = \begin{bmatrix} 0 & A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$(L^{-1}R)^4 = \begin{bmatrix} A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$(L^{-1}R)^5 = \begin{bmatrix} 0 & A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$\vdots$$

Now $\Lambda_0^{-1} = L^{-1}$ and $\Lambda_m^{-1} = (-1)^m(L^{-1}R)^m L^{-1}$ thus

$$\Lambda_0^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

$$\Lambda_1^{-1} = \begin{bmatrix} 0 & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & 0 \end{bmatrix}$$

$$\Lambda_2^{-1} = \begin{bmatrix} A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}$$

$$\Lambda_3^{-1} = \begin{bmatrix} 0 & -A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$\vdots$$

Let $\Lambda^{-1} = B$. Summing elements,

$$\begin{aligned} B_{11} &= A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1} + \cdots \\ -B_{12} &= A_{11}^{-1}A_{12}A_{22}^{-1} + A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1} \\ &\quad + A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}A_{22}^{-1} + \cdots \end{aligned}$$

etc. for the elements of B or

$$\Lambda^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Finally,

$$B_{11} = (A_{11}^{-1}) \left[\sum_{n=0}^{\infty} \{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n \right]$$

or

$$B_{11} = \left[\sum_{n=0}^{\infty} \{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n \right] (A_{11}^{-1})$$

are equivalent representations of B_{11} (right inverse and left inverse respectively).

Similarly for the right and left inverse for B_{12}

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)$$

$$= -\left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1}A_{12}A_{22}^{-1})$$

$$B_{21} = -(A_{22}^{-1}A_{21}A_{11}^{-1})\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right)$$

$$= -\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}A_{21}A_{11}^{-1})$$

$$B_{22} = (A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)$$

$$= \left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1})$$

the second form being the left inverse in each case. Thus for left inverse multiplication $\Lambda^{-1}\Lambda = I$, we have

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = I$$

Hence

$$\begin{bmatrix} B_{11}A_{11} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} \\ B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} \end{bmatrix}$$

so that we must have as a verification

$$B_{11}A_{11} + B_{12}A_{21} = I \tag{1}$$

$$B_{11}A_{12} + B_{12}A_{22} = 0 \tag{2}$$

$$B_{21}A_{11} + B_{22}A_{21} = 0 \tag{3}$$

$$B_{21}A_{12} + B_{22}A_{22} = I \tag{4}$$

From (1)

$$\left[\left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1})\right]A_{11} + \left[-\left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1}A_{12}A_{22}^{-1})\right]A_{21}$$

$$= \sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n - \sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^{n+1}$$

$$= \sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n - \sum_{m=1}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^m$$

$$= I + \sum_{n=1}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n - \sum_{m=1}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^m$$

$$= I$$

Now from (2)

$$B_{11}A_{12} + B_{12}A_{22} = 0$$

$$\left[\left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1})\right]A_{12} + \left[-\left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1}A_{12}A_{22}^{-1})\right]A_{22}$$

$$= \left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1}A_{12}) - \left(\sum_{n=0}^{\infty}\{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n\right)(A_{11}^{-1}A_{12}) = 0$$

From (3)

$$B_{21}A_{11} + B_{22}A_{21} = 0$$

$$\left[-\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}A_{21}A_{11}^{-1})\right]A_{11}$$

$$+ \left[\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1})\right]A_{21}$$

$$= -\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}A_{21}) + \left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}A_{21}) = 0$$

From (4)

$$B_{21}A_{12} + B_{22}A_{22} = I$$

$$\left[-\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}A_{21}A_{11}^{-1})\right]A_{12} + \left[\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1})\right]A_{22}$$

$$= -\left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^{n+1}\right) + \left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right) = I$$

proving relations $1-4$. Noting that the first B_{ij} formula is used for right inverse multiplications.

$$AA^{-1} = I$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

We must have

$$A_{11}B_{11} + A_{12}B_{21} = I \tag{5}$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \tag{6}$$

$$A_{21}B_{11} + A_{22}B_{21} = 0 \tag{7}$$

$$A_{21}B_{12} + A_{22}B_{22} = I \tag{8}$$

From (5) verifying

$$(A_{11})\left[(A_{11}^{-1})\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right)\right] + (A_{12})\left[-(A_{22}^{-1}A_{21}A_{11}^{-1})\right.$$

$$\left.\times\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right)\right]$$

$$= \left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right) - \left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{n+1}\right) = I$$

From (6)

$$A_{11}B_{12} + A_{12}B_{22} = 0$$

$$(A_{11})\left[-(A_{11}^{-1}A_{12}A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)\right]$$

$$+ (A_{12})\left[(A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)\right]$$

$$= -(A_{12}A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right) + (A_{12}A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right) = 0$$

From (7) verifying

$$A_{21}B_{11} + A_{22}B_{21} = 0$$

we have

$$(A_{21})\left[(A_{11}^{-1})\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right)\right] + (A_{22})\left[-(A_{22}^{-1}A_{21}A_{11}^{-1})\right.$$

$$\left.\times\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right)\right]$$

$$= (A_{21}A_{11}^{-1})\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right) - (A_{21}A_{11}^{-1})\left(\sum_{n=0}^{\infty}\{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n\right) = 0$$

Similarly from (8)

$$A_{21}B_{12} + A_{22}B_{22} = I$$

$$(A_{21})\left[-(A_{11}^{-1}A_{12}A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)\right]$$

$$+ (A_{22})\left[(A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right)\right]$$

$$= -\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{n+1} + \sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n = I$$

In beginning with an $n \times n$ matrix we write it as a partitioned matrix

$$\Lambda = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Where A_{11} is $p \times p$, A_{22} is $q \times q$, A_{12} is $p \times q$, A_{21} is $q \times p$ where $p + q = n$. If n is even, we may choose $p = q = n/2$ and we have shown

$$\Lambda^{-1}_{(n \times n)} \equiv B_{(n \times n)} = \begin{bmatrix} B_{11(n \times p)} & B_{12(p \times q)} \\ B_{21(q \times p)} & B_{22(q \times q)} \end{bmatrix}$$

$$I_{(n \times n)} = \begin{bmatrix} I_{(p \times p)} & 0_{(p \times q)} \\ 0_{(q \times p)} & I_{(q \times q)} \end{bmatrix}$$

where I is the identity matrix and 0 the zero matrix. Consider the 2×2 case where the entries are numbers. We have

$$\Lambda = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$A_{11}^{-1} = a_{11}^{-1}$$

$$A_{22}^{-1} = a_{22}^{-1}$$

which is now easily evaluated or one can immediately write the final sum as we have seen previously[2, 4]. It is interesting to note that for the 1×1 case if we write $A = (a_{11}) = L + R$ where $L = a_{11}$ and $R = \alpha_{11}$

$$L^{-1} = 1/a_{11} \qquad R = \alpha_{11}$$

$$L^{-1}R = \alpha_{11}/a_{11}$$

$$A^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\alpha_{11}}{a_{11}}\right)^n \frac{1}{a_{11}}$$

$$= \frac{1}{a_{11}} - \frac{\alpha_{11}}{a_{11}^2} + \frac{\alpha_{11}\alpha_{11}}{a_{11}^3} - \cdots$$

which appears to be a possible means of evaluating $\langle A^{-1} \rangle$ when A has a stochastic component α_{11}.

However returning to the immediate problem, we now consider a 3×3 case.

$$\Lambda = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\Lambda - 1 = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

$$A_{11}^{-1} = 1/a_{11}$$

To find A_{22}^{-1} we partition A_{22}

$$A_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} A_{22,11} & A_{22,12} \\ A_{22,21} & A_{22,22} \end{bmatrix}$$

Now

$$A_{22}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{22}^{-1} R_{22})^n L_{22}^{-1}$$

where

$$L_{22} = \begin{bmatrix} A_{22,11} & 0 \\ 0 & A_{22,22} \end{bmatrix} \qquad R_{22} = \begin{bmatrix} 0 & A_{22,12} \\ A_{22,21} & 0 \end{bmatrix}$$

$$L_{22}^{-1} = \begin{bmatrix} A_{22,11}^{-1} & 0 \\ 0 & A_{22,22}^{-1} \end{bmatrix} \qquad L_{22}^{-1} R_{22} = \begin{bmatrix} 0 & A_{22,11}^{-1} A_{22,12} \\ A_{22,22}^{-1} A_{22,21} & 0 \end{bmatrix}$$

where $A_{22,11}^{-1} = 1/a_{22}$ and $A_{22,22}^{-1} = 1/a_{33}$.

4 × 4 Case:

$$\Lambda = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \qquad L^{-1}R = \begin{bmatrix} 0 & A_{11}^{-1} A_{12} \\ A_{22}^{-1} A_{21} & 0 \end{bmatrix}$$

$$A_{11} = \left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11,11} & A_{11,12} \\ \hline A_{11,21} & A_{11,22} \end{array}\right]$$

$$A_{22} = \left[\begin{array}{c|c} a_{33} & a_{34} \\ \hline a_{43} & a_{44} \end{array}\right] \left[\begin{array}{c|c} A_{22,11} & A_{22,12} \\ \hline A_{22,21} & A_{22,22} \end{array}\right]$$

$$A_{11}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{11}^{-1} R_{11})^n L_{11}^{-1}$$

$$A_{22}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{22}^{-1} R_{22})^n L_{22}^{-1}$$

$$L_{11} = \begin{bmatrix} A_{11,11} & 0 \\ 0 & A_{11,22} \end{bmatrix} \qquad R_{11} = \begin{bmatrix} 0 & A_{11,12} \\ A_{11,21} & 0 \end{bmatrix}$$

$$L_{11}^{-1} = \begin{bmatrix} A_{11,11}^{-1} & 0 \\ 0 & A_{11,12}^{-1} \end{bmatrix} \qquad L_{11}^{-1} R_{11} = \begin{bmatrix} 0 & A_{11,11}^{-1} A_{11,12} \\ A_{11,22}^{-1} A_{11,21} & 0 \end{bmatrix}$$

$$L_{22} = \begin{bmatrix} A_{22,11} & 0 \\ 0 & A_{22,22} \end{bmatrix} \qquad R_{22} = \begin{bmatrix} 0 & A_{22,12} \\ A_{22,21} & 0 \end{bmatrix}$$

$$L_{22}^{-1} = \begin{bmatrix} A_{22,11}^{-1} & 0 \\ 0 & A_{22,22}^{-1} \end{bmatrix} \qquad L_{22}^{-1} R_{22} = \begin{bmatrix} 0 & A_{22,11}^{-1} A_{22,12} \\ A_{22,22}^{-1} A_{22,21} & 0 \end{bmatrix}$$

$$A_{11,11}^{-1} = 1/a_{11}$$

$$A_{11,22}^{-1} = 1/a_{22}$$

$$A_{22,11}^{-1} = 1/a_{33}$$

$$A_{22,22}^{-1} = 1/a_{44}$$

Numerical example: 4×4 case:

$$\Lambda = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

We choose a symmetric banded matrix (whose elements are diagonal matrices) to make calculations easier. We note this is Case I where $|L| > |R|$ and

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

(When Λ is unequally partitioned, A_{11} and A_{22} are square and $|L| = |A_{11}A_{22}| = |A_{11}|\cdot|A_{22}|$ and $|R| = |A_{21}A_{12}| = |A_{21}|\cdot|A_{12}|$. With unequal partitioning, A_{11}, A_{22} are still square so the above holds for $|L|$ but for $|R|$, we have only $|R| = |A_{21}\cdot A_{12}|$).

$$\Lambda = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $\Lambda = L + R$ (case I)

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A_{22} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \qquad A_{12} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A_{11}^{-1} = \begin{bmatrix} \dfrac{a_{22}}{a_{11}a_{22} - a_{21}a_{12}} & \dfrac{-a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ \dfrac{-a_{21}}{a_{11}a_{22} - a_{21}a_{12}} & \dfrac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$$

$$A_{22}^{-1} = \begin{bmatrix} \dfrac{a_{44}}{a_{33}a_{44} - a_{43}a_{34}} & \dfrac{-a_{34}}{a_{33}a_{44} - a_{43}a_{34}} \\ \dfrac{-a_{43}}{a_{33}a_{44} - a_{43}a_{34}} & \dfrac{a_{33}}{a_{33}a_{44} - a_{43}a_{34}} \end{bmatrix}$$

using formulas developed in [2, 4] for the 2×2 matrix to which the decomposition series for the inverse will converge.

Now using the inversion theorem,

$$B_{11} = B_{22} = (A_{11}^{-1})(I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1})^{-1}$$

Calculating $A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Hence

$$I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/4 & 0 \\ 0 & 3/4 \end{bmatrix}$$

and

$$(I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1})^{-1} = \begin{bmatrix} 4/3 & 0 \\ 0 & 4/3 \end{bmatrix}$$

Finally

$$(A_{11}^{-1})(I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1})^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4/3 & 0 \\ 0 & 4/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

so that

$$B_{11} = B_{22} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$A_{11} = A_{22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A_{21} = A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} B_{12} = B_{21} &= -(A_{11}^{-1}A_{12}A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1} \\ &= -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4/3 & 0 \\ 0 & 4/3 \end{bmatrix} \\ &= -\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \end{aligned}$$

Hence

$$B_{12} = B_{21} = \begin{bmatrix} -1/3 & 0 \\ 0 & -1/3 \end{bmatrix}$$

Thus

$$\Lambda^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix}$$

Multiplying

$$\Lambda = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

from the left by B or Λ^{-1} we have

$$\Lambda^{-1}\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or the 4 $\times$ 4 identity matrix composed of

$$\begin{bmatrix} I_{(2\times 2)} & 0_{(2\times 2)} \\ 0_{(2\times 2)} & I_{(2\times 2)} \end{bmatrix}$$

so we have verified the inverse is correct. It is interesting to note the characteristic rapid convergence generally observed with the decomposition series in the following chart, where S_n is the sum of the geometric series for n terms of $\sum_{n=0}^{\infty} (A_{12}A_{22}^{-1}A_{21}A_{11}^{-1})^n$ and ϕ_n is the n term approximant to Λ^{-1}

n	S_n	ϕ_n
0	1	
1	.250	1.00
2	.0625	1.25
3	.015625	1.3125
4	.00390625	1.328125
5	.0009765625	1.33203125
6	.0002441406	1.333007813
7	.0000610352	1.333251953
8	.0000152586	1.333312988
9	.0000038147	1.333328247
10	.0000009537	1.333332062
$\vdots$		
∞	0	1.333333333 . . . = 4/3

The error by $n = 5$ is less than 0.1% and decreases as shown for higher n. Consider now again a Case I example but with an unsymmetrical partitioning as shown

$$\Lambda = \left[\begin{array}{c|ccc} 2 & 0 & 1 & 0 \\ \hline 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$A_{22} = \left[\begin{array}{c|cc} 2 & 0 & 1 \\ \hline 0 & 2 & 0 \\ 1 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} A_{22,11} & A_{22,12} \\ \hline A_{22,21} & A_{22,22} \end{array}\right]$$

$$A_{22,22} = \left[\begin{array}{c|c} 2 & 0 \\ \hline 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} A_{22,22,11} & A_{22,22,12} \\ \hline A_{22,22,21} & A_{22,22,22} \end{array}\right]$$

$$A_{11}^{-1} = 1/2$$

$$A_{22,11}^{-1} = 1/2$$

$$A_{22,22,11}^{-1} = 1/2$$

$$A_{22,22,22} = 1/2$$

since these are 1 $\times$ 1 or scalar cases their inverses are trivial. Now decompose Λ into

$L + R$ where

$$L = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

$$A_{11}^{-1} = 1/2$$

$$A_{22} = \begin{bmatrix} A_{22,11} & A_{22,12} \\ A_{22,21} & A_{22,22} \end{bmatrix}$$

$$L_{22} = \begin{bmatrix} A_{22,11} & 0 \\ 0 & A_{22,22} \end{bmatrix} \qquad R_{22} = \begin{bmatrix} 0 & A_{22,12} \\ A_{22,21} & 0 \end{bmatrix}$$

$$L_{22}^{-1} = \begin{bmatrix} A_{22,11}^{-1} & 0 \\ 0 & A_{22,22}^{-1} \end{bmatrix}$$

$$A_{22,11}^{-1} = 1/2$$

$$A_{22,22} = \begin{bmatrix} A_{22,22,11} & A_{22,22,12} \\ A_{22,22,21} & A_{22,22,22} \end{bmatrix}$$

$$L_{22,22} = \begin{bmatrix} A_{22,22,11} & 0 \\ 0 & A_{22,22,22} \end{bmatrix}$$

$$R_{22,22} = \begin{bmatrix} 0 & A_{22,22,12} \\ A_{22,22,12} & 0 \end{bmatrix}$$

$$A_{22} = \left[\begin{array}{c|cc} 2 & 0 & 1 \\ \hline 0 & 2 & 0 \\ 1 & 0 & 2 \end{array}\right] = \begin{bmatrix} A_{22,11} & A_{22,12} \\ A_{22,21} & A_{22,22} \end{bmatrix}$$

$$A_{22,11}^{-1} = 1/2 \qquad A_{22,22}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A_{22,12} = (0 \quad 1) \qquad A_{22,21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_{22}^{-1} = B_{22} = \begin{bmatrix} B_{22,11} & B_{22,12} \\ B_{22,21} & B_{22,22} \end{bmatrix}$$

$$B_{22,11} = A_{22,11}^{-1}\{I - A_{22,12}A_{22,22}^{-1}A_{22,21}A_{22,11}^{-1}\}^{-1}$$

$$B_{22,12} = -(A_{22,11}^{-1}A_{22,12}A_{22,22}^{-1})\{I - A_{22,21}A_{22,11}^{-1}A_{22,12}A_{22,22}^{-1}\}^{-1}$$

$$B_{22,21} = (A_{22,22}^{-1}A_{22,21}A_{22,11}^{-1})\{I - A_{22,12}A_{22,22}^{-1}A_{22,21}A_{22,11}^{-1}\}^{-1}$$

$$B_{22,22} = (A_{22,22}^{-1})\{I - A_{22,21}A_{22,11}^{-1}A_{22,12}A_{22,22}^{-1}\}^{-1}$$

Hence

$$\begin{aligned} B_{22,12} &= -(1/2)\,(0 \quad 1) \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/3 \end{bmatrix} \\ &= -(1/2)\,(0 \quad 1) \begin{bmatrix} 1/2 & 0 \\ 0 & 2/3 \end{bmatrix} \\ &= -(1/2)\,(0 \quad 2/3) = (0 \quad -1/3) \end{aligned}$$

Similarly

$$B_{22,21} = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1/2)\,(4/3)$$

$$= \begin{bmatrix} 0 \\ -1/3 \end{bmatrix}$$

$$B_{22,11} = (1/2)\left\{1 - (0 \quad 1)\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(1/2)\right\}^{-1}$$

$$= 2/3$$

$$B_{22,22} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$B_{22,11} = 2/3$$

$$B_{22,22} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$B_{22,12} = (0 \quad -1/3)$$

$$B_{22,21} = \begin{bmatrix} 0 \\ -1/3 \end{bmatrix}$$

$$B_{22} = \left[\begin{array}{c|c} B_{22,11} & B_{22,12} \\ \hline B_{22,21} & \cdot\ B_{22,22} \end{array}\right]$$

$$= \left[\begin{array}{c|cc} 2/3 & 0 & -1/3 \\ \hline 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{array}\right]$$

Verifying

$$A_{22} \cdot B_{22} = \begin{bmatrix} (2) & (0 \quad 1) \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} (2/3) & (0 \quad -1/3) \\ \begin{bmatrix} 0 \\ -1/3 \end{bmatrix} & \begin{bmatrix} 1/2 & 0 \\ 0 & 2/3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1) & (0 \quad 0) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} (I) & 0 \\ (0) & (I) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Now

$$L_{22,22}^{-1} = \begin{bmatrix} A_{22,22,11}^{-1} & 0 \\ 0 & A_{22,22,22}^{-1} \end{bmatrix}$$

$$A_{22,22,11}^{-1} = 1/2$$

$$A_{22,22,22}^{-1} = 1/2$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_{22}^{-1} = \left[\begin{array}{c|cc} 1/2 & 0 & 0 \\ \hline 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{array}\right] \qquad R_{22} = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$A_{11}^{-1} = 1/2 \qquad A_{22}^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$$

$$A_{12} = (0 \quad 1 \quad 0) \qquad A_{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$B_{11} = A_{11}^{-1}\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1}$$

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

$$B_{21} = -(A_{22}^{-1}A_{21}A_{11}^{-1})\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1}$$

$$B_{22} = A_{22}^{-1}\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

We compute

$$A_{12}A_{22}^{-1}A_{21}A_{11}^{-1} = (0 \quad 1 \quad 0)\begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}(1/2) = 1/4$$

$$\{I - 1/4\}^{-1} = 4/3$$

$$A_{21}A_{11}^{-1}A_{12}A_{22}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot (1/2) \cdot (0 \quad 1 \quad 0) \cdot \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$$

$$\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$$B_{11} = (1/2)\,(4/3) = 2/3$$

$$B_{12} = -(1/2)\,(0 \quad 1 \quad 0)\begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0 \quad -1/3 \quad 0)$$

$$B_{21} = -\begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}(1/2)\,(4/3) = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 2/3 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$$

Hence

$$\Lambda^{-1} = B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix}$$

Checking the result we have

$$\Lambda\Lambda^{-1} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \equiv I$$

thus the inverse is correct.

Example: 5×5 matrix partitioned as shown. Again we choose a symmetric banded case to minimize computation. We have selected A_{22} to be a previously calculated case.

$$\Lambda = \left[\begin{array}{c|cccc} 2 & 0 & 1 & 0 & 1 \\ \hline 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$A_{11}^{-1} = 1/2$$

$$A_{22}^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix}$$

$$B_{11} = A_{11}^{-1}\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1}$$

$$B_{22} = A_{22}^{-1}\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

$$B_{21} = -(A_{22}^{-1}A_{21}A_{11}^{-1})\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1}$$

Computing $A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}$ we have

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot (1/2) \cdot (0 \quad 1 \quad 0 \quad 1) \cdot \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 1/6 \\ 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 1/6 \end{bmatrix}$$

Then computing $[I - A_{21}A_{11}^{-1}A_{12}A_{22}]^{-1}$ we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/4 & 0 & 5/4 \end{bmatrix}$$

Similarly $A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}$ is given by

$$[0 \quad 1 \quad 0 \quad 1] \cdot \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} [1/2] = 1/3$$

Hence $\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1} = 3/2$

$$B_{11} = A_{11}^{-1}\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1} = 3/4$$

$$B_{22} = A_{22}^{-1}\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

$$= \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/4 & 0 & 5/4 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 3/4 & 0 & -1/4 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/4 & 0 & 3/4 \end{bmatrix}$$

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1}$$

$$= -(1/2) \cdot (0 \quad 1 \quad 0 \quad 1) \cdot \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/4 & 0 & 5/4 \end{bmatrix}$$

$$= (0 \quad -1/4 \quad 0 \quad -1/4)$$

$$B_{21} = -(A_{22}^{-1})(A_{21})(A_{11}^{-1})\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1}$$

$$= -\begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot (1/2) \cdot (3/2)$$

$$= \begin{bmatrix} 0 \\ -1/4 \\ 0 \\ -1/4 \end{bmatrix}$$

Since

$$\Lambda^{-1} = B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

we can substitute to get

$$B = \left[\begin{array}{c|cccc} 3/4 & 0 & -1/4 & 0 & -1/4 \\ \hline 0 & 2/3 & 0 & -1/3 & 0 \\ -1/4 & 0 & 3/4 & 0 & -1/4 \\ 0 & -1/3 & 0 & 2/3 & 0 \\ -1/4 & 0 & -1/4 & 0 & 3/4 \end{array}\right] = \Lambda^{-1}$$

and it can be verified that $\Lambda\Lambda^{-1} = I_{(5\times 5)}$ or

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We observe that the partitioning adaptation of the decomposition method yields the previously derived inverse matrix for the 2 × 2 case in [2, 4]. Let

$$\Lambda = \left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

Then $A_{11}^{-1} = 1/a_{11}$, $A_{22}^{-1} = 1/a_{22}$

$$\Lambda^{-1} = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$\begin{aligned} B_{11} &= (1/a_{11})\left(1 - a_{12}\frac{1}{a_{22}}a_{21}\frac{1}{a_{11}}\right)^{-1} \\ &= a_{22}(a_{11}a_{22} - a_{12}a_{21})^{-1} \\ B_{12} &= (1/a_{11})a_{12}(1/a_{22})\left(1 - a_{21}\frac{1}{a_{11}}a_{12}\frac{1}{a_{22}}\right)^{-1} \\ &= -a_{12}(a_{11}a_{22} - a_{21}a_{12})^{-1} \\ B_{21} &= -(1/a_{22})a_{21}(1/a_{11})\left(1 - a_{12}\frac{1}{a_{22}}a_{21}\frac{1}{a_{11}}\right)^{-1} \\ &= -a_{21}(a_{11}a_{22} - a_{12}a_{21})^{-1} \\ B_{22} &= (1/a_{22})\left(1 - a_{21}\frac{1}{a_{11}}a_{12}\frac{1}{a_{22}}\right)^{-1} \\ &= a_{11}(a_{11}a_{22} - a_{21}a_{12})^{-1} \end{aligned}$$

which checks the result[2, 4]

$$\Lambda^{-1} = \begin{bmatrix} a_{22}/\alpha & -a_{12}/\alpha \\ -a_{21}/\alpha & a_{11}/\alpha \end{bmatrix}$$

with $\alpha = a_{11}a_{22} - a_{21}a_{12}$.

For the 3×3 case

$$\Lambda = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$A_{11} = a_{11}$$

$$A_{12} = (a_{12}a_{13})$$

$$A_{21} = \begin{vmatrix} a_{21} \\ a_{31} \end{vmatrix}$$

$$A_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A_{11}^{-1} = 1/a_{11}$$

$$A_{22}^{-1} = \begin{bmatrix} a_{33}/\alpha & -a_{23}/\alpha \\ -a_{32}/\alpha & a_{22}/\alpha \end{bmatrix}$$

where $\alpha = a_{22}a_{33} - a_{32}a_{23}$ from the previous example and the formulas are now applicable to determine the B_{ij}

$$\Lambda^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$$

$$L^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

$$L^{-1}R = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

$$B_{11} = B_{22} = (A_{11}^{-1})(I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1})^{-1}$$

etc.

For the 4×4 case

$$\Lambda = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A_{12} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \qquad A_{22} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$$

$$A_{11}^{-1} = \begin{bmatrix} a_{22}/\alpha & -a_{12}/\alpha \\ -a_{21}/\alpha & a_{11}/\alpha \end{bmatrix}$$

where $\alpha = a_{11}a_{22} - a_{21}a_{12}$

$$A_{22}^{-1} = \begin{bmatrix} a_{44}/\beta & -a_{34}/\beta \\ -a_{43}/\beta & a_{33}/\beta \end{bmatrix}$$

where $\beta = a_{33}a_{44} - a_{43}a_{34}$ from the 2×2 example and the formulas for B_{ij} can now be applied.

We observe in passing some interesting patterns for symmetric banded matrices without investigation of possible restrictions.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 & 0 \\ 0 & 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & 2/3 \end{bmatrix}$$

i.e. that the divisor is the sum of the row elements.

To invert an 8×8 matrix

$$A = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where the elements A_{11}, A_{12}, A_{21}, A_{22} are now 4×4

$$L^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \qquad R = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

Now

$$A_{11}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{11}^{-1} R_{11})^n L_{11}^{-1}$$

$$L_{11}^{-1} = \begin{bmatrix} A_{11,11}^{-1} & 0 \\ 0 & A_{11,22}^{-1} \end{bmatrix}$$

$$R_{11} = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix}$$

$$A_{22}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{22}^{-1} R_{22})^n L_{22}^{-1}$$

$$L_{22}^{-1} = \begin{bmatrix} A_{22,11}^{-1} & 0 \\ 0 & A_{22,22}^{-1} \end{bmatrix}$$

$$R_{22} = \begin{bmatrix} 0 & 0 & a_{57} & a_{58} \\ 0 & 0 & a_{67} & a_{68} \\ a_{75} & a_{76} & 0 & 0 \\ a_{85} & a_{86} & 0 & 0 \end{bmatrix}$$

$$A_{11,11}^{-1} = \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix} \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \right\}^n \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix}$$

$$A_{11,22}^{-1} = \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{bmatrix} 1/a_{33} & 0 \\ 0 & 1/a_{44} \end{bmatrix} \begin{bmatrix} 0 & a_{34} \\ a_{43} & 0 \end{bmatrix} \right\}^n \begin{bmatrix} 1/a_{33} & 0 \\ 0 & 1/a_{44} \end{bmatrix}$$

$$A_{22,11}^{-1} = \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{bmatrix} 1/a_{55} & 0 \\ 0 & 1/a_{66} \end{bmatrix} \begin{bmatrix} 0 & a_{56} \\ a_{65} & 0 \end{bmatrix} \right\}^n \begin{bmatrix} 1/a_{55} & 0 \\ 0 & 1/a_{66} \end{bmatrix}$$

$$A_{22,22}^{-1} = \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{bmatrix} 1/a_{77} & 0 \\ 0 & 1/a_{88} \end{bmatrix} \begin{bmatrix} 0 & a_{78} \\ a_{87} & 0 \end{bmatrix} \right\}_n \begin{bmatrix} 1/a_{77} & 0 \\ 0 & 1/a_{88} \end{bmatrix}$$

SUMMARY OF PROCEDURE

1) Partition the given matrix Λ into

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

(where the partitioning need not be symmetrical and the partitioned components need not be square.)

2) Partition A_{11} and A_{22} into

$$A_{11} = \left[\begin{array}{c|c} A_{11,11} & A_{11,12} \\ \hline A_{11,21} & A_{11,22} \end{array}\right]$$

$$A_{22} = \left[\begin{array}{c|c} A_{22,11} & A_{22,12} \\ \hline A_{22,21} & A_{22,22} \end{array}\right]$$

3) Partition $A_{11,11}$, $A_{11,22}$, $A_{22,11}$, and $A_{22,22}$ into

$$A_{11,11} = \left[\begin{array}{c|c} A_{11,11,11} & A_{11,11,12} \\ \hline A_{11,11,21} & A_{11,11,22} \end{array}\right]$$

$$A_{11,12} = \left[\begin{array}{c|c} A_{11,22,11} & A_{11,22,12} \\ \hline A_{11,22,21} & A_{11,22,22} \end{array}\right]$$

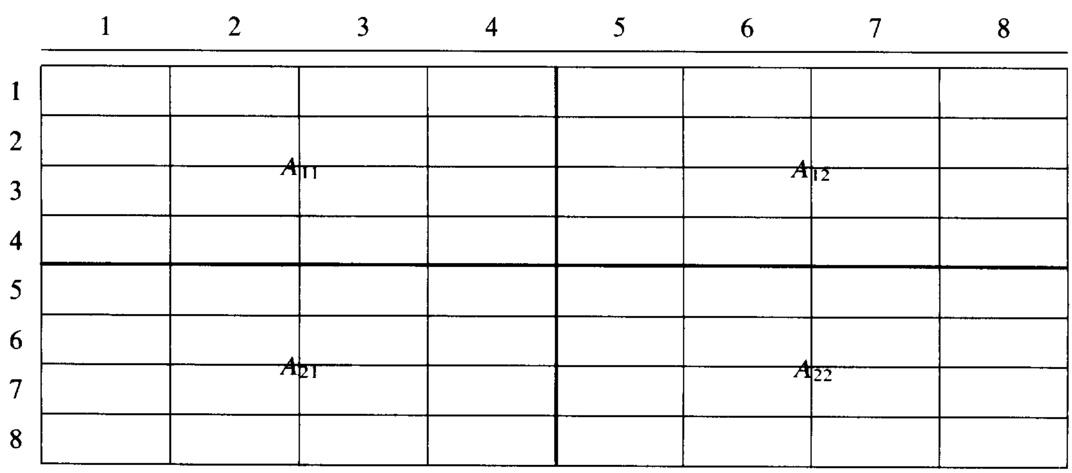
$$A_{22,11} = \left[\begin{array}{c|c} A_{22,11,11} & A_{22,11,12} \\ \hline A_{22,11,21} & A_{22,11,22} \end{array}\right]$$

$$A_{22,22} = \left[\begin{array}{c|c} A_{22,22,11} & A_{22,22,12} \\ \hline A_{22,22,21} & A_{22,22,22} \end{array}\right]$$

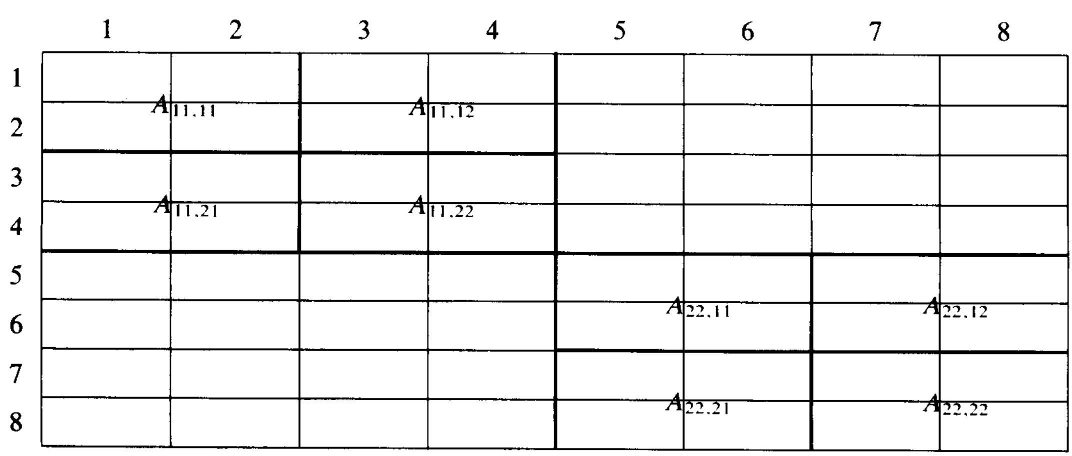
$$\vdots$$

The following diagrams may be helpful in systematizing the procedure.

PARTITION—STAGE I



PARTITION—STAGE II



PARTITION—STAGE III

	1	2	3	4	5	6	7	8
1	$A_{11,11,11}$	$A_{11,11,12}$						
2	$A_{11,11,21}$	$A_{11,11,22}$						
3			$A_{11,22,11}$	$A_{11,22,12}$				
4			$A_{11,22,21}$	$A_{11,22,22}$				
5					$A_{22,11,11}$	$A_{22,11,12}$		
6					$A_{22,11,21}$	$A_{22,11,22}$		
7							$A_{22,22,11}$	$A_{22,22,12}$
8							$A_{22,22,21}$	$A_{22,22,22}$

L DECOMPOSITION—STAGE I

	1	2	3	4	5	6	7	8
1								
2								
3			A_{11}				0	
4								
5								
6								
7			0				A_{22}	
8								

R DECOMPOSITION—STAGE I

	1	2	3	4	5	6	7	8
1								
2								
3			0				A_{12}	
4								
5								
6								
7			A_{12}				0	
8								

L DECOMPOSITION—STAGE II

	1	2	3	4	5	6	7	8
1	$A_{11,11}$		0					
2								
3	0		$A_{11,22}$					
4								
5					$A_{22,11}$		0	
6								
7					0		$A_{22,22}$	
8								

R DECOMPOSITION—STAGE II

	1	2	3	4	5	6	7	8
1	0		$A_{11,12}$					
2								
3	$A_{11,21}$		0					
4								
5					0		$A_{22,12}$	
6								
7					$A_{22,21}$		0	
8								

L DECOMPOSITION—STAGE III

	1	2	3	4	5	6	7	8
1	$A_{11,11,11}$	0						
2	0	$A_{11,11,22}$						
3			$A_{11,22,11}$	0				
4			0	$A_{11,22,22}$				
5					$A_{22,11,11}$	0		
6					0	$A_{22,11,22}$		
7							$A_{22,22,11}$	0
8							0	$A_{22,22,22}$

R DECOMPOSITION—STAGE III

	1	2	3	4	5	6	7	8
1	0	$A_{11.11.12}$						
2	$A_{11.11.21}$	0						
3			0	$A_{11.22.12}$				
4			$A_{11.22.21}$	0				
5					0	$A_{22.11.12}$		
6					$A_{22.11.21}$	0		
7							0	$A_{22.22.12}$
8							$A_{22.22.21}$	0

Recapitulating the essential formulae

$$\Lambda = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is $p \times p$, A_{12} is $p \times q$, A_{21} is $q \times p$, A_{22} is $q \times q$ where $p + q = n$. If n is even, we may choose $p = q = n/2$.

$$\Lambda^{-1}B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$, B_{12} is $p \times q$, B_{21} is $q \times p$, B_{22} is $q \times q$ and the elements of B are given by

$$B_{11} = (A_{11}^{-1}) \left(\sum_{n=0}^{\infty} \{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n \right) \quad \text{for the right inverse}$$

$$= \left(\sum_{n=0}^{\infty} \{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n \right) (A_{11}^{-1}) \quad \text{for the left inverse}$$

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1}) \left(\sum_{n=0}^{\infty} \{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n \right) \quad \text{for the right inverse}$$

$$= - \left(\sum_{n=0}^{\infty} \{A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^n \right) (A_{11}^{-1}A_{12}A_{22}^{-1}) \quad \text{for the left inverse}$$

$$B_{21} = -(A_{22}^{-1}A_{21}A_{11}^{-1}) \left(\sum_{n=0}^{\infty} \{A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^n \right) \quad \text{for the right inverse}$$

$$- \left(\sum_{n=0}^{\infty} \{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n \right) (A_{22}^{-1}A_{21}A_{11}^{-1}) \quad \text{for the left inverse}$$

$$B_{22} = (A_{22}^{-1})\left(\sum_{n=0}^{\infty}\{A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^n\right) \quad \text{for the right inverse}$$

$$= \left(\sum_{n=0}^{\infty}\{A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^n\right)(A_{22}^{-1}) \quad \text{for the left inverse}$$

which we may represent accurately with a few terms. (Proofs were given earlier in this paper.) Finally we can write analytic summations and the theorem

INVERSION THEOREM. For the matrices Λ and Λ^{-1} or B given above, the elements of B are:

$$B_{11} = (A_{11}^{-1})\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1} \quad \text{for the left inverse and}$$

$$B_{11} = \{I - A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^{-1}(A_{11}^{-1}) \quad \text{for the right inverse}$$

$$B_{12} = -(A_{11}^{-1}A_{12}A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1} \quad \text{for the left inverse and}$$

$$B_{12} = -\{I - A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\}^{-1}(A_{11}^{-1}A_{12}A_{22}^{-1}) \quad \text{for the right inverse}$$

$$B_{21} = -(A_{22}^{-1}A_{21}A_{11}^{-1})\{I - A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\}^{-1} \quad \text{for the left inverse}$$

and

$$B_{21} = -\{I - A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^{-1}(A_{22}^{-1}A_{21}A_{11}^{-1}) \quad \text{for the right inverse}$$

$$B_{22} = -(A_{22}^{-1})\{I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}\}^{-1} \quad \text{for the left inverse and}$$

$$B_{22} = \{I - A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}\}^{-1}(A_{22}^{-1})$$

Of course one can compute the inverse by writing $\Lambda = L + R$ where L contains only diagonal elements so L^{-1} is obtained for either Case I or Case II, writing for R all the elements of Λ with the diagonal elements missing and (if $|L| > |R|$) calculating the decomposition series. For small matrices especially if R has small elements this is sufficient. (In the limiting case where all elements of R are zeros, $\Lambda^{-1} \equiv L^{-1}$ and is trivial.) However for matrices which are not sparse and especially for large matrices, the partitioned decomposition is far better and becomes indispensable for extremely large matrices with hundreds or thousands of elements. Inaccuracies arise of course from the need to round off numbers during the computational process which is dependent on the computer capability. Considerations of ill-conditioned matrices should apply here also so the elements of Λ^{-1} may be unstable if the determinant of Λ, for example is small. Finally, we emphasize that this method applies to sparse as well as dense matrices.

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